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Some unsolved problems in the theory of homogeneous spaces[1]

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Abstract

In this paper we formulate some problems of the theory of homogeneous spaces. In particular, we discuss a behaviour of the Ricci solitons under the conformal deformations of the initial metric. In addition, we present some problems the theory of geodesically orbital spaces.

In recent years intensively investigated Ricci solitons, i.e. Riemannian manifolds (M, g), which satisfy to the differential equation:

$$r(g) = \Lambda g + L_X(g),$$

where $r(g)$ is the Ricci tensor of the metric g, Λ is a real constant, $L_X(g)$ is the Lie derivative of g in the direction of a full vector field X.

Note that in the case where $L_X(g) = 0$, we have the usual Einstein equation for the Riemannian manifolds and the corresponding solitons are called trivial. Thus, Ricci solitons are a natural generalization of the Riemannian manifolds with metric of Einstein, and the theory of solitons is a generalization of the theory of Riemann manifolds with Einstein metric. Einstein manifolds and methods of their construction are well known (see., for example, the surveys [1, 2]). In particular, a deformation of the initial Riemannian metric is one of the ways of constructing new Einstein metrics. We note that conformal deformation is one of them : $g^1 = e^{2f}g$, where $f = f(x)$ is a smooth function on M. Under this deformation the Ricci tensor and the Lie derivative are changed by the formulas:

$$r^1 = r - (n-2)(D\,df - df \circ df) + (\Delta f - (n-2)|df|^2)g$$

$$L_X(e^{2f}g) = \left\langle X, grade^{2f} \right\rangle g + e^{2f} L_X(g),$$

where Df is the gradient, Δf is the Laplacian and Ddf is the Hessian of f with respect to g. It is clear, therefore, that there is an opportunity to construct new Ricci solitons from existing. In particular, assuming that $M = G/H$ is a homogeneous space, and $(M = G/H, g)$ is homogeneous, or algebraic Ricci soliton, we obtain a system of algebraic and differential equations defining the new Ricci solitons on locally conformally homogeneous spaces.

Problem 1. *Construct new Ricci solitons with the help of the conformal deformation of the initial metric.*

Since a homogeneous space (M, ρ) is geodesically complete, there arises the problem on the behavior of geodesic curves on such spaces, their closure, and on their self-intersection. The following theorem is known in this direction.

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Theorem 1. *(see [3,4]). Geodesics on homogeneous spaces are merely closed curves or unclosed curves without self-intersections.*

Moreover, the following theorem is proved in the work [4] of M.V. Mechsheryakov.

Theorem 2. *Geodesic curves of a left-invariant metric on a connected and simply connected nilpotent Lie group are not closed.*

The following two problems arise in a natural way:

Problem 2. *(A. Besse). Find homogeneous Riemannian manifolds all of whose geodesics are closed.*

Problem 3. *Describe homogeneous (pseudo)Riemannian manifolds all of whose geodesics are unclosed.*

Problem 4. *Describe a behavior of the geodesic curves on homogeneous pseudo-Riemannian manifolds.*

For the first time, the Besse problem was considered in the class of normal homogeneous spaces, i.e., those spaces $(G/H, \rho)$ whose homogeneous Riemannian metric ρ is obtained from the $Ad(G)$-invariant inner product of a Lie group G under the projection $\pi : G \to G/H$. The following theorem was proved in [5].

Theorem 3. *(see [5]). Let $(G/H, \rho)$ be a simply connected, normal homogeneous Riemannian spaceall of whose geodesics are closed. Then $(G/H, \rho)$ is isometric to a compact symmetric space of rank 1 (CSROS: S^n, CP^k, HP^m, and CaP^2).*

Later on, by using purely topological methods, the following theorem was proved in [5] for arbitrary homogeneous Riemannian manifolds.

Theorem 4. *(see [5]). A simply connected homogeneous Riemannian manifold all of whose geodesicsare closed and have the same length is isometric to a CSROS.*

Simultaneously, a geometric proof of this theorem having no requirement on the lengths of geodesics was given [6, 7].

Theorem 5. *(see [6, 7]). A simply connected Riemannian manifold all of whose geodesics are closed is isometric to a CSROS.*

The main idea of the proof of Theorem 5 is as follows. If the structure of $(G/H, \rho)$ is complicated, then we seek a flat totally geodesic torus T in $M = G/H$ whose irrational winding is unclosed. Then a finite list of manifolds remains, which is examined step-by-step.

Definition 1. *A geodesic γ of a Riemannian manifold (M, ρ) is said to be homogeneous if it is an orbit of a one-parameter subgroup $g(t)$ of $Isom(M, \rho)$.*

The following theorem is known.

Theorem 6. *(see [8]). Every homogeneous Riemannian manifold has at least one homogeneous geodesic passing through any point given in advance.*

As is conventional, a geodesic γ is said to be maximal if it is not the restriction of any other geodesic.

Definition 2. *A homogeneous manifold $(G/H, \rho)$ is called a geodesically orbital space if all of its maximal geodesics are homogeneous.*

Remark 1. *Naturally reductive and, in particular, normal homogeneous spaces are geodesically orbital spaces.*

There naturally arises the problem on the existence of a geodesically orbital space different from a naturally reductive space. The first such example was constructed by A. Kaplan [9]. This example is the six-dimensional nilpotent Lie group with two-dimensional center (one of the generalized Heisenberg groups) equipped with a certain left-invariant metric.

The class of *weakly symmetric* spaces is closely related to the class of geodesically orbital spaces.

Definition 3. *A Riemannian manifold M is said to be a weakly symmetric space if for every pair of points p, q of M, there exists an isometry of M interchanging the points p and q.*

It is clear that any symmetric space is weakly symmetric and naturally reductive. Also, geodesic spheres in symmetric spaces of rank 1 are weakly symmetric. Note that there exist weakly symmetric spaces which are not even naturally reductive. For example, geodesic spheres in the Cayley projective plane CaP^2 are such spaces. J. Berndt, O. Kowalski, and L. Vanhecke obtained the following result in [10].

Theorem 7. *(see [10]). Every weakly symmetric space M is geodesically orbital.*

Many examples of weakly symmetric spaces were constructed by W. Ziller in [11]. The geodesically orbital spaces of dimension $\leqslant 6$ were classified by O. Kowalski and L. Vanhecke in [12]. It turns out that all geodesically orbital spaces of dimension $\leqslant 5$ are naturally reductive. At the same time, in the case of dimension equal to 6, there exist three- and two-parameter families of geodesically orbital spaces that are not naturally reductive. Among these families, there is the compact symmetric homogeneous space $SO(5)/U(2)$ having a two-parameter family of invariant metrics.

The structure of geodesically orbital spaces was also studied by Gordon in [13], where the case of nilpotent Lie groups with left-invariant Riemannian metric was studied in detail.

Among recent works, we can mention the work D. Alekseevsky and A. Arvanitoyeorgos [14] devoted to metrics with homogeneous geodesics on flag manifolds.

At the same time, the following problems remain unsolved.

Problem 5. *Classify all geodesically orbital (pseudo)Riemannian spaces.*

Problem 6. *Find Jacobi fields and conjugate points along geodesic curves on the geodesically orbital (pseudo)Riemannian spaces.*

In concluding we mention the works about other problems in the theory of homogeneous spaces [15–24].

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