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On the Schouten-Weyl tensor of 3-dimensional metric Lie groups[1]

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Abstract

The main purpose of this paper is to investigate the Schouten–Weyl tensor on the three-dimensional Lie groups with left-invariant Lorenzian metrics. The left-invariant Lorentzian metrics on the three-dimensional Lie groups with squared length zero Schouten–Weyl tensor are studied. Moreover, the three-dimensional metric Lie groups with almost harmonic (i.e. with zero curl and divergence) Schouten–Weyl tensor are investigated. In addition, the question about the harmonicity of contraction of the Schouten–Weyl tensor is considered.

1. Introduction

The fundamental monography [1] give us a survey about the Riemannian metrics with harmonic Weyl tensor. In dimension three, the Weyl tensor is trivial, therefore we consider the Schouten–Weyl tensor (the Cotton tensor), which plays a role of the Weyl tensor. The Schouten–Weyl tensor was investigated in [2] for the case of left-invariant Lorentzian metrics on the three-dimensional Lie groups. The previous paper was a continuation of J. Milnor's paper [3] about the left-invariant Riemannian metrics on three-dimensional Lie groups. A classification of the three-dimensional metric Lie algebras of the Lie groups with a left-invariant Riemannian metric and trivial divergence and curl of the Schouten–Weyl tensor is given in [4].

2. Basic notation and facts

Let G be a Lie group, $\{\mathfrak{g}, [\cdot\cdot]\}$ is a corresponding Lie algebra. We denote a left-invariant Lorentzian metric on G by $\langle\cdot, \cdot\rangle$, Levi-Civita connection by ∇, curvature tensor and Ricci tensor by R and r, which defined by the following

$$R(X, Y)Z = [\nabla_Y, \nabla_X]Z + \nabla_{[X,Y]}Z, \quad r(X, Y) = \mathrm{tr}(Z \mapsto R(X, Z)Y).$$

Further we fix a basis $\{E_1, E_2, \ldots, E_n\}$ of left-invariant vector fields in $\mathfrak{g}$ and set

$$[E_i, E_j] = c_{ij}^k E_k, \quad \langle E_i, E_j \rangle = g_{ij}, \tag{1}$$

where $\{c_{ij}^k\}$ are a structure constants of the Lie algebra and $\{g_{ij}\}$ is a metric tensor.

We consider the one-dimensional curvature tensor A and Schouten–Weyl tensor SW, which defined by the formulas

$$A_{ik} = \frac{1}{n-2}\left(r_{ik} - \frac{\rho g_{ik}}{2(n-1)}\right), \quad SW_{ijk} = A_{ij,k} - A_{ik,j},$$

where ρ is a scalar curvature, $A_{ij,k}$ are a covariant derivatives of A_{ij}.

The squared length of the Schouten–Weyl tensor is defined by

$$\|SW\|^2 = SW_{ijk}SW^{ijk}.$$

Definition 1. *A tensor T is isotropic, if $\|T\|^2 = 0$ and $T \neq 0$.*

Definition 2. *A tensor $T_{i_1 \ldots i_p}$ is harmonic (cf. [5, p.43]), if:*

1. *$T_{i_1 \ldots i_p}$ is antisymmetric,*

2. *$\mathrm{curl}(T_{i_1 i_2 \cdots i_p}) = 0$ or $T_{i_1 i_2 \cdots i_p, t} = T_{t i_2 \cdots i_p, i_1} + T_{i_1 t \cdots i_p, i_2} + \cdots + T_{i_1 i_2 \cdots t, i_p},$*

3. *$\mathrm{div}(T_{i_1 i_2 \cdots i_p}) = g^{i_1 t} T_{i_1 \ldots i_p, t} = 0.$*

We note that the antisymmetric part $SW_{[ijk]}$ and symmetric part $SW_{(ijk)}$ of the Schouten–Weyl tensor vanish. Thus, condition 1) of Definition 2 is not satisfied by the Schouten–Weyl tensor.

Definition 3. *A tensor $T_{i_1 \ldots i_p}$ is said to be almost harmonic if:*

1. *$\mathrm{curl}(T_{i_1 i_2 \cdots i_p}) = 0,$*

2. *$\mathrm{div}(T_{i_1 i_2 \cdots i_p}) = 0.$*

For the Schouten–Weyl tensor SW_{ijk} we introduce the divergence of type I and II by

$$\mathrm{div}_1(SW) = g^{it} SW_{ijk,t}, \quad \text{and} \quad \mathrm{div}_2(SW) = g^{jt} SW_{ijk,t}.$$

Let $V = V^i E_i$ be a left-invariant vector field, which is identified with the vector $\{V^k\}$. Let w_{ij} be the contraction of the Schouten–Weyl tensor SW_{kij} with the vector field $\{V^k\}$, i.e.

$$w_{ij} = V^k SW_{kij}. \tag{2}$$

Since the Schouten–Weyl tensor SW_{kij} is antisymmetric with respect to i and j, the tensor w_{ij} is antisymmetric. The covariant derivatives of w_{ij} have the form

$$w_{ij,k} = w_{lj}\Gamma^l_{ki} + w_{il}\Gamma^l_{kj}.$$

The curl and divergence of the tensor w_{ij} are expressed as

$$\mathrm{curl}(w) = w_{ij,t} - w_{tj,i} - w_{it,j},$$
$$\mathrm{div}(w) = g^{it} w_{ij,t}.$$

The length of the vector field $\{V^k\}$ squared is expressed as

$$\|V\|^2 = g_{ij} V^i V^j, \tag{3}$$

where g_{ij} is the metric tensor of Lorentzian signature.

Together with the arbitrary vector fields we consider the harmonic vector fields.

Definition 4. *A vector field $\{V^i\}$ is called harmonic if:*

1. *$\mathrm{curl}(V) = V^i_{,j} - V^j_{,i} = 0$,*

2. *$\mathrm{div}(V) = V^i_{,i} = 0,$*

where the covariant derivatives of $\{V^i\}$ are found by

$$V^i_{,k} = -V^l \Gamma^i_{lk}.$$

Further classification results for three-dimensional Lorentzian Lie groups was obtained in [2, 6].

Theorem 1. *Let G be a three-dimensional unimodular Lie group with left-invariant Lorentzian metric. Then there exists a pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$, such that the metric Lie algebra of G is one of the following:*

1.

$$\mathcal{A}_1 : \quad \begin{aligned} [e_1, e_2] &= \lambda_3 e_3, \\ [e_1, e_3] &= -\lambda_2 e_2, \\ [e_2, e_3] &= \lambda_1 e_1, \end{aligned}$$

with e_1 timelike;

2.

$$\mathcal{A}_2 : \quad \begin{aligned} [e_1, e_2] &= (1 - \lambda_2)\, e_3 - e_2, \\ [e_1, e_3] &= e_3 - (1 + \lambda_2)\, e_2, \\ [e_2, e_3] &= \lambda_1 e_1, \end{aligned}$$

with e_3 timelike;

3.

$$\mathcal{A}_3 : \quad \begin{aligned} [e_1, e_2] &= e_1 - \lambda e_3, \\ [e_1, e_3] &= -\lambda e_2 - e_1, \\ [e_2, e_3] &= \lambda_1 e_1 + e_2 + e_3, \end{aligned}$$

with e_3 timelike;

4.

$$\mathcal{A}_4 : \quad \begin{aligned} [e_1, e_2] &= \lambda_3 e_2, \\ [e_1, e_3] &= -\beta e_1 - \alpha e_2, \\ [e_2, e_3] &= -\alpha e_1 + \beta e_2, \end{aligned}$$

with e_1 timelike and $\beta \neq 0$.

Remark 1. *There are exactly six nonisomorphic three-dimensional unimodular Lie algebras and the corresponding types of three-dimensional unimodular Lie groups (see [3]). All of them are listed in the Table 1 together with conditions on structure constants for which the Lie algebra has this type. If there is a "−" in the Table 1 at the intersection of the row, corresponding to the Lie algebra, and the column, corresponding to the type, then it means that this type of the basis is impossible for given Lie algebra. For the case of Lie algebra $\mathcal{A}_1$ we give only the signs of the triple $(\lambda_1, \lambda_2, \lambda_3)$ up to reorder and sign change.*

Remark 2. *We note that similar bases was also constructed by G. Calvaruso, L. A. Cordero and P. E. Parker in [7, 8].*

3. The isotropy of Schouten–Weyl tensor

In the paper [2] the problem of existence of pseudo-Riemannian metrics, for which the squared length of the Schouten–Weyl tensor is zero and some of the components of the Schouten–Weyl tensor are not zero sumultaneously has been tasked. This problem has been solved for the cases $\mathcal{A}_1$ and $\mathcal{A}_4$, and for nonunimodular three-dimensional Lie algebras. This paper presents a solutions to the problem for the cases $\mathcal{A}_2$ and $\mathcal{A}_3$.

Table 1

Three-dimensional unimodular Lie algebras

Lie algebra	Restrictions on the structure constants			
	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$
$su(2)$	$(+,+,+)$	$-$	$-$	$-$
$sl(2,\mathbb{R})$	$(+,+,-)$	$\lambda_1 \neq 0, \lambda_2 \neq 0$	$\lambda \neq 0$	$\lambda_3 \neq 0$
$e(2)$	$(+,+,0)$	$-$	$-$	$-$
$e(1,1)$	$(+,-,0)$	$\lambda_1 = 0, \lambda_2 \neq 0$ or $\lambda_1 \neq 0, \lambda_2 = 0$	$\lambda = 0$	$\lambda_3 = 0$
h	$(+,0,0)$	$\lambda_1 = 0, \lambda_2 = 0$	$-$	$-$
$\mathbb{R}^3$	$(0,0,0)$	$-$	$-$	$-$

Table 2

Three-dimensional unimodular metric Lie algebras $\mathcal{A}_2$ and $\mathcal{A}_3$ with the isotropic Schouten–Weyl tensor

Lie algebra	Restrictions on the structure constants	Type of Lie algebra
$\mathcal{A}_2$	$\lambda_1 = 0, \lambda_2 \neq 0$	$e(1,1)$
	$\lambda_1 = \lambda_2 \neq 0$	$sl(2,\mathbb{R})$
$\mathcal{A}_3$	$\lambda \neq 0$	

Theorem 2. *Let G be an unimodular three-dimensional Lie group with left-invariant Lorentzian metric, Lie algebra $\mathcal{A}_2$ or $\mathcal{A}_3$ and isotropic Schouten–Weyl tensor. Then the Lie algebra of group G is isomorphic to $e(1,1)$ or $sl(2,\mathbb{R})$, and the restrictions on structural constants are contained in the Table 2.*

Proof. We consider the case $\mathcal{A}_2$. Let be $\{E_1, E_2, E_3\}$ a basis, which was listed in the Theorem 2. Calculating the components of the Schouten–Weyl tensor with help of the previously presented formulas, we see, that the non-trivial components of the Schouten–Weyl tensor have the form

$$
\begin{aligned}
SW_{132} &= -\lambda_1^3 + \lambda_1^2\lambda_2, \\
SW_{221} &= SW_{331} = -\frac{1}{2}\lambda_1^2 - 2\lambda_2\lambda_1 + 4\lambda_2^2, \\
SW_{231} &= -\frac{1}{2}\lambda_1^2 - 2\lambda_2\lambda_1 + 4\lambda_2^2 - \frac{1}{2}\lambda_1^3 + \frac{1}{2}\lambda_1^2\lambda_2, \\
SW_{321} &= -\frac{1}{2}\lambda_1^2 - 2\lambda_2\lambda_1 + 4\lambda_2^2 + \frac{1}{2}\lambda_1^3 - \frac{1}{2}\lambda_1^2\lambda_2,
\end{aligned}
\tag{4}
$$

and the squared length of the Schouten–Weyl tensor is equal to

$$
\|SW\|^2 = -3\lambda_1^4(\lambda_1 - \lambda_2)^2.
$$

The formula shows that equality to the zero is achieved if: $\lambda_1 = 0$ or $\lambda_1 = \lambda_2$. Thus the Schouten–Weyl tensor is not trivial if and only if $\lambda_1 = 0, \lambda_2 \neq 0$ or $\lambda_1 = \lambda_2 \neq 0$.

Now we consider the case $\mathcal{A}_3$. Let be $\{E_1, E_2, E_3\}$ is a basis, which was listed in the Theorem 2. As in the case of $\mathcal{A}_2$, we calculate the components of the Schouten–Weyl

tensor

$$SW_{121} = -SW_{131} = SW_{232} = -SW_{332} = \frac{3}{2}\lambda^2,$$
$$SW_{221} = -SW_{231} = -SW_{321} = SW_{331} = -6\lambda. \tag{5}$$

Thus the squared length of the Schouten–Weyl tensor is trivial for any value of the structure constant λ, and the Schouten–Weyl tensor can be zero only if $\lambda \equiv 0$. $\square$

Remark 3. *The case of Lie algebra $\mathcal{A}_1$ or $\mathcal{A}_4$ and case of non-unimodular Lie algebras has been considered in [2].*

4. Almost harmonicity of the Schouten–Weyl tensor

In this part we investigate the three-dimensional metric Lie groups with zero curl and divergence of the Schouten–Weyl tensor.

Theorem 3. *Let G be an unimodular three-dimensional Lie group with left-invariant Lorentzian metric and Lie algebra $\mathcal{A}_2$ or $\mathcal{A}_3$. Then $\mathrm{div}_1(SW) \equiv 0$. If moreover $\mathrm{curl}(SW) = 0$, i.e. the Schouten–Weyl tensor is almost harmonic, then the Schouten–Weyl tensor is trivial (so $\mathrm{div}_2(SW) = 0$), and the Lie algebra of group G is contained in the Table 3.*

Table 3

The three-dimensional unimodular metric Lie algebras $\mathcal{A}_2$ or $\mathcal{A}_3$ with trivial Schouten–Weyl tensor

Lie algebra	Restrictions on the structure constants	Type of Lie algebra
$\mathcal{A}_2$	$\lambda_1 = \lambda_2 = 0$	h
$\mathcal{A}_3$	$\lambda = 0$	$e(1,1)$

Proof. We consider the case of Lie algebra $\mathcal{A}_2$. Direct calculations show that $\mathrm{div}_1(SW) \equiv 0$.

Further we assume that the Schouten–Weyl tensor is almost harmonic, i.e. $\mathrm{curl}(SW) = 0$, so we have the system of equations:

$$-2\lambda_2\lambda_1^2 - \lambda_1^3 + 16\lambda_2^2\lambda_1 - 16\lambda_2^3 = 0,$$
$$\left(\lambda_1^2 + 4\lambda_1\lambda_2 - 8\lambda_2^2\right)(\lambda_1 - 2\lambda_2) = 0,$$
$$-\lambda_1^3 + 2\lambda_2\lambda_1^2 + 8\lambda_2^2\lambda_1 + 3\lambda_1^4 - 3\lambda_2\lambda_1^3 = 0, \tag{6}$$
$$-7\lambda_1^3 + 2\lambda_2\lambda_1^2 + 8\lambda_2^2\lambda_1 = 0,$$
$$7\lambda_1^3 - 2\lambda_2\lambda_1^2 - 3\lambda_1^4 + 3\lambda_2\lambda_1^3 - 8\lambda_2^2\lambda_1 = 0.$$

Solving the system of equations (6) for the structure constants of the Lie algebra $\mathcal{A}_2$, we obtain the solution $\lambda_1 = \lambda_2 = 0$. It is immediately verified that for this solution the Schouten–Weyl tensor (4) is trivial.

Next we consider the case of Lie algebra $\mathcal{A}_3$. Direct calculations show that in this case we have $\mathrm{div}_1(SW) \equiv 0$.

Let the Schouten–Weyl tensor satisfies the condition $\mathrm{curl}(SW) = 0$. In this case, we obtain the following system of equations:

$$\lambda = 0, \quad \lambda^2 = 0, \quad \lambda^3 = 0,$$
$$16\lambda + \lambda^3 = 0, \quad 16\lambda^2 - \lambda^3 = 0, \quad \lambda^3 - 6\lambda = 0, \quad \lambda^3 + 6\lambda = 0. \tag{7}$$

Solving the system of equations (7) for the structure constants of the Lie algebra $\mathcal{A}_3$, we obtain the solution $\lambda = 0$. It is easy to verify, that for this solution the Schouten–Weyl tensor (5) is trivial. $\qquad\square$

Remark 4. *The case of Lie algebra $\mathcal{A}_1$ or $\mathcal{A}_4$ and case of non-unimodular Lie algebras has been considered in [9].*

5. The harmonicity of contraction of the Schouten–Weyl tensor

In this section we consider the question about the harmonicity of contraction of the Schouten–Weyl tensor, which was defined by (2), through the Schouten–Weyl tensor SW and some vector field V.

Theorem 4. *Let G be an unimodular three-dimensional Lie group with left-invariant Lorentzian metric, Lie algebra $\mathcal{A}_2$ or $\mathcal{A}_3$ and w is a harmonic tensor. Then the restrictions on structure constants of the Lie algebra of group G and components of vector field $\{V^k\}$ are contained in the Table 4.*

Proof. First we consider the case of Lie algebra $\mathcal{A}_2$. Using (2), we find the nontrivial components of the tensor w_{ij}

$$w_{12} = \frac{1}{2}V^2\lambda_1^2 - 2V^2\lambda_1 - 6V^2 - 2V^2\lambda_2^2 + 2V^3 + 2V^3\lambda_2^2 - 2V^3\lambda_2\lambda_1 +$$

$$+ \frac{1}{2}V^3\lambda_2\lambda_1^2 + \frac{1}{2}V^3\lambda_1^3 + \frac{1}{2}V^3\lambda_1^2 - 6V^3\lambda_2 - 2V^3\lambda_2^3,$$

$$w_{13} = -2V^2\lambda_2^2 - 2V^2\lambda_2\lambda_1 + \frac{1}{2}V^2\lambda_2\lambda_1^2 - \frac{1}{2}V^2\lambda_1^3 - \frac{1}{2}V^2\lambda_1^2 - 6V^2\lambda_2 -$$

$$- 2V^2\lambda_2^3 - 2V^2 - \frac{1}{2}V^3\lambda_1^2 + 2V^3\lambda_1 + 6V^3 + 2V^3\lambda_2^2,$$

$$w_{23} = -V^1(\lambda_1^3 + \lambda_1^2 + 4\lambda_2^2 + 4).$$

Computing the curl and divergence of the tensor w_{ij}, we see that the equalities $\operatorname{curl}(w) \equiv 0$ and $\operatorname{div}(w) \equiv 0$ are equivalent to the system of equations

$$-V^1(\lambda_1^3 + \lambda_1^2 + 4\lambda_2^2 + 4)\lambda_1 = 0,$$

$$4V^2\lambda_2\lambda_1 + V^2\lambda_2\lambda_1^3 - V^2\lambda_2^2\lambda_1^2 + 4V^2\lambda_2^2\lambda_1 + 16V^2 - 16V^3 + 20V^2\lambda_2^2 - 8V^3\lambda_2^2 -$$

$$- V^3\lambda_1^3 + V^2\lambda_1^3 + 8V^2\lambda_2^3 + 4V^2\lambda_2^4 - 4V^3\lambda_1 + 16V^2\lambda_2 + 4V^2\lambda_1 = 0, \qquad (8)$$

$$-4V^3\lambda_2\lambda_1 + 4V^3\lambda_2^2\lambda_1 - V^3\lambda_2^2\lambda_1^2 - V^3\lambda_2\lambda_1^3 - 16V^2 + 16V^3 - 8V^2\lambda_2^2 + 20V^3\lambda_2^2 +$$

$$+ V^3\lambda_1^3 - 8V^3\lambda_2^3 - V^2\lambda_1^3 + 4V^3\lambda_2^4 - 16V^3\lambda_2 + 4V^3\lambda_1 - 4V^2\lambda_1 = 0.$$

Solving the system of equations (8) for the structure constants of the Lie algebra and the components of vector field $\{V^k\}$, we define Lie algebras and the corresponding directions, for which the tensor w_{ij} is harmonic. We obtain the following solutions:

1. $V = (V^1, V^2, V^3)$, $V^i \in \mathbb{R}$, $\lambda_1 = -2$, $\lambda_2 = 0$;

2. $V = (0, V^2, V^3)$, $\quad V^2, V^3 \in \mathbb{R} \setminus \{0\}$, $\quad \lambda_1 = \frac{1}{6}\left(F(V^2, V^3) + \frac{H(V^2, V^3)}{F(V^2, V^3)} - (V^2 + V^3)^2\right)$,
   $\lambda_2 = \frac{(V^3)^2 - (V^2)^2}{2V^2V^3}$, $V^2 \neq \left(-7 \pm 4\sqrt{3} + 2\sqrt{24 + 14\sqrt{3}}\right)V^3$, where

$$F(x, y) = \left(P(x, y) + 6\sqrt{6(x^2 + y^2)(x - y)^4 Q(x, y)}\right)^{1/3},$$

$$P(x, y) = 53x^6 - 150x^5y + 507x^4y^2 - 308x^3y^3 + 507x^2y^4 - 150xy^5 + 53y^6,$$

$$Q(x, y) = 13x^6 - 22x^5y + 163x^4y^2 - 52x^3y^3 + 163x^2y^4 - 22xy^5 + 13y^6,$$

$$H(x, y) = x^4 + 28x^3y + 6x^2y^2 + 28xy^3 + y^4;$$

Table 4

The three-dimensional unimodular metric Lie algebras $\mathcal{A}_2$ or $\mathcal{A}_3$ with harmonic contraction of the Schouten–Weyl tensor

Restrictions on the structure constants	Type of Lie algebra	Vector field
Lie algebra $\mathcal{A}_2$		
$\lambda_1 = -2,\ \lambda_2 = 0$	$e(1,1)$	$V = (V^1, V^2, V^3),\ V^i \in \mathbb{R}$ is harmonic.
$\lambda_i = L_i(V^2, V^3)$	$sl(2,\mathbb{R})$ or $e(1,1)$	$V = (0, V^2, V^3),\ V^2, V^3 \in \mathbb{R} \setminus \{0\}$, $V^2 \neq \left(-7 \pm 4\sqrt{3} + 2\sqrt{24 + 14\sqrt{3}}\right) V^3$ is harmonic.
$\lambda_2 = 0,\ \lambda_1 \neq 0$	$e(1,1)$	$V = (0, V^2, V^2),\ V^2 \in \mathbb{R}$ is harmonic.
$\lambda_1 = 0,\ \lambda_2 \neq 0$	$e(1,1)$	$V = (V^1, 0, 0),\ V^1 \in \mathbb{R}$
Lie algebra $\mathcal{A}_3$		
$\lambda = 0$	$e(1,1)$	$V = (0, -V^3, V^3),\ V^3 \in \mathbb{R}$ is harmonic.
$\lambda = L_3$	$sl(2,\mathbb{R})$	$V = (V^1, V^2, V^3),\ V^3 \in \mathbb{R}$, $V^1 = f(\lambda)V^3,\ V^2 = h(\lambda)V^3$

where

$$L_1(x,y) = \frac{1}{6}\left(F(x,y) + \frac{H(x,y)}{F(x,y)} - (x+y)^2\right), \quad L_2(x,y) = \frac{y^2 - x^2}{2xy},$$

$$F(x,y) = \left(P(x,y) + 6\sqrt{6(x^2+y^2)(x-y)^4 Q(x,y)}\right)^{1/3},$$

$$P(x,y) = 53x^6 - 150x^5y + 507x^4y^2 - 308x^3y^3 + 507x^2y^4 - 150xy^5 + 53y^6,$$

$$Q(x,y) = 13x^6 - 22x^5y + 163x^4y^2 - 52x^3y^3 + 163x^2y^4 - 22xy^5 + 13y^6,$$

$$H(x,y) = x^4 + 28x^3y + 6x^2y^2 + 28xy^3 + y^4,$$

$$L_3 = \pm\frac{1}{4}\sqrt{\sqrt{6\left(136643 + 512A + \frac{104960}{A} + \frac{85821417}{B}\right)} - 483 - B} \approx \pm 89.072,$$

$$A = \left(3763 + 6\sqrt{154029}\right)^{1/3} \approx 18.289,$$

$$B = \frac{1}{A}\sqrt{409929A - 3072A^2 - 629760} \approx 565.076,$$

$$f(\lambda) = \frac{1}{8192}\lambda\left(440\lambda^6 + 9047\lambda^4 - 47248\lambda^2 + 30400\right),$$

$$h(\lambda) = -\frac{1}{2048}\left(88\lambda^6 + 2219\lambda^4 + 4322\lambda^2 - 2112\right).$$

3. $V = (0, V^2, V^2),\ V^2 \in \mathbb{R},\ \lambda_1 \in \mathbb{R} \setminus \{0\},\ \lambda_2 = 0$;

4. $V = (V^1, 0, 0),\ \lambda_1 = 0,\ \lambda_2 \in \mathbb{R} \setminus \{0\},\ V^1 \in \mathbb{R}$.

Let $\{V^k\}$ be harmonic. Computing the curl and divergence of the vector field $\{V^k\}$, we see that the equalities $\operatorname{curl}(V) \equiv 0$ and $\operatorname{div}(V) \equiv 0$ are equivalent to the equation:

$$V^1(2 + \lambda_1) = 0. \tag{9}$$

Solving the system of equations (8) and (9), we obtain, that the directions 1–3 are harmonic and the direction 4 isn't harmonic.

Now we consider the case of Lie algebra $\mathcal{A}_3$. Using (2), we find the nontrivial components of the tensor w_{ij}

$$w_{12} = -\frac{11}{2}V^1\lambda^2 + 4V^1 + 2V^2\lambda - \lambda^3V^3 + 2V^3\lambda,$$

$$w_{13} = \frac{3}{2}V^1\lambda^2 - 4V^1 - 2\lambda^3V^2 + 6V^2\lambda - 2V^3\lambda,$$

$$w_{23} = -\lambda^3V^1 + 4V^1\lambda + \frac{3}{2}V^2\lambda^2 - 4V^2 - 4V^3 + \frac{11}{2}V^3\lambda^2.$$

Computing the curl and divergence of the tensor w_{ij}, we see that the equalities $\mathrm{curl}(w) \equiv 0$ and $\mathrm{div}(w) \equiv 0$ are equivalent to the system of equations:

$$\begin{aligned}
-16V^1 + 22V^1\lambda^2 - \lambda^3V^2 - 16V^3\lambda - 2\lambda^4V^1 + 13\lambda^3V^3 &= 0, \\
-\lambda^3V^1 + 4\lambda^4V^2 - 15V^2\lambda^2 - 7V^3\lambda^2 + 8V^2 + 8V^3 &= 0, \\
-16V^1\lambda + 13\lambda^3V^1 - 7V^2\lambda^2 + 2\lambda^4V^3 - 15V^3\lambda^2 + 8V^2 + 8V^3 &= 0.
\end{aligned} \tag{10}$$

Solving this system of equations for the structure constants of the Lie algebra and the components of vector field $\{V^k\}$, we obtain the following solutions:

1. $V = (0, -V^3, V^3)$, $V^3 \in \mathbb{R}$, $\lambda = 0$;

2. $V = (V^1, V^2, V^3)$, $V^3 \neq 0$ and

$$V^1 = \frac{1}{8192}V^3\lambda(440\lambda^6 + 9047\lambda^4 - 47248\lambda^2 + 30400),$$

$$V^2 = -\frac{1}{2048}V^3(88\lambda^6 + 2219\lambda^4 + 4322\lambda^2 - 2112),$$

$$\lambda = \pm\frac{1}{4}\sqrt{\sqrt{6\left(136643 + 512A + \frac{104960}{A} + \frac{85821417}{B}\right)} - 483 - B} \approx \pm89.072;$$

$$A = (3763 + 6\sqrt{154029})^{1/3} \approx 18.289,$$

$$B = \frac{1}{A}\sqrt{409929A - 3072A^2 - 629760} \approx 565.076.$$

Let $\{V^k\}$ be harmonic. Computing the curl and divergence of the vector field $\{V^k\}$, we see that the equalities $\mathrm{curl}(V) \equiv 0$ and $\mathrm{div}(V) \equiv 0$ are equivalent to the equation:

$$-\lambda V^1 + 2V^2 + 2V^3 = 0. \tag{11}$$

Solving the system of equations (10) and (11), we obtain, that the direction 1 is harmonic and the direction 2 isn't harmonic. $\qquad\square$

Remark 5. *A similar theorems for the contraction of the Schouren-Weyl tensor in the cases $\mathcal{A}_1$ and $\mathcal{A}_4$ and also in the case of non-unimodular Lie algebras has been proved in [9].*

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