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On three-dimensional locally homogeneous manifolds with vectorial torsion and zero curvature tensor[1]

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Abstract

This paper is devoted to solving the problem of studying locally homogeneous (pseudo)Riemannian manifolds with metric connection with vectorial torsion, the curvature tensor of which is zero.

Keywords: Metric connection, vectorial torsion, locally homogeneous manifold, curvature tensor

1. Preliminaries

Let (M, g) be a (pseudo)Riemannian manifold. Define a metric connection ∇ on M by the formula

$$\nabla_X Y = \nabla^g_X Y + g\left(X, Y\right) V - g\left(V, Y\right) X,$$

where V is a fixed vector field, X, Y are arbitrary vector fields, ∇^g is the Levi-Civita connection on M. This connection is called *a* metric connection with vectorial torsion [1].

Let R be a curvature tensor of a metric connection ∇. It is determined by the equality

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z.$$

As known, the following conditions are hold

1) $R(X, Y)Z = -R(Y, X)Z$ (for any linear connection);
2) $R(X, Y, Z, T) = -R(X, Y, T, Z)$ (for any metric connection).

Also we have

Theorem 1. *[2] A necessary and sufficient condition that the Ricci tensor of the metric connection ∇ to be symmetric is that the (0,4) curvature tensor R of the connection ∇ satisfies one of the following conditions:*

1. $R(X, Y, Z, U) = R(Z, U, X, Y)$,
2. $R(X, Y, Z, U) + R(Y, Z, X, U) + R(Z, X, Y, U) = 0$,

where $R(X, Y, Z, V) = g(R(X, Y)Z, U)$.

Let $M = G/H$ be a locally homogeneous (pseudo)Riemannian manifold; $\mathfrak{g}$ be a Lie algebra of isometry group G, $\mathfrak{h}$ be a Lie algebra of isotropy subgroup H, $\mathfrak{m}$ be a complement to $\mathfrak{h}$ in $\mathfrak{g}$. Denote $\dim \mathfrak{h} = h$ and $\dim \mathfrak{m} = m$. One can fix a basis $\{e_1, \ldots, e_h, u_1, u_2, \ldots, u_m\}$ of algebra $\mathfrak{g}$, where $\{e_i\}$ and $\{u_i\}$ are bases of $\mathfrak{h}$ and $\mathfrak{m}$ respectively. Denote

$$[u_i, u_j]_{\mathfrak{m}} = c^k_{ij} u_k, \quad [u_i, u_j]_{\mathfrak{h}} = C^k_{ij} e_k, \quad [e_i, u_j]_{\mathfrak{m}} = \bar{c}^k_{ij} u_k,$$

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where c_{ij}^k, C_{ij}^k and $\bar{c}_{ij}^k$ are arrays of appropriate sizes. Components of the Levi-Civita connection ∇^g can be expressed in terms of structural constants and components of the metric tensor:

$$(\Gamma^g)_{ij}^k = \frac{1}{2}\left(c_{ij}^k + g^{sk}c_{sj}^l g_{il} + g^{sk}c_{si}^l g_{jl}\right), \quad \left(\bar{\Gamma}^g\right)_{ij}^k = \frac{1}{2}\bar{c}_{ij}^k - \frac{1}{2}g^{sk}\bar{c}_{is}^l g_{jl},$$

where $\nabla^g_{u_i} u_j = (\Gamma^g)_{ij}^k u_k$, $\nabla^g_{e_i} u_j = \left(\bar{\Gamma}^g\right)_{ij}^k u_k$; $\{g^{ij}\}$ is inverse of $\{g_{ij}\}$.

Let invariant vector $V \in \mathfrak{m}$, then components of metric connection ∇ with vectorial torsion are defined by:

$$\Gamma_{ij}^k = (\Gamma^g)_{ij}^k + g_{ij}V^k - V^s g_{sj}\delta_i^k, \quad \bar{\Gamma}_{ij}^k = \left(\bar{\Gamma}^g\right)_{ij}^k,$$

where $\nabla_{u_i} u_j = \Gamma_{ij}^k u_k$, $\nabla_{e_i} u_j = \bar{\Gamma}_{ij}^k u_k$.

Components of the curvature tensor R can be calculated via formula:

$$R_{ijks} = \left(\Gamma_{ik}^l \Gamma_{jl}^p - \Gamma_{jk}^l \Gamma_{il}^p + c_{ij}^l \Gamma_{lk}^p + C_{ij}^l \bar{\Gamma}_{lk}^p\right) g_{ps}$$

or

$$\begin{aligned} R_{ijks} = &\frac{1}{4}(c_{ik}^r + g^{tr}c_{tki} + g^{tr}c_{tik} + 2g_{ik}V^r - 2\delta_i^r V_k) \cdot (c_{jrs} + c_{srj} + c_{sjr} + 2g_{jr}V_s - 2g_{js}V_r) - \\ &-\frac{1}{4}(c_{jk}^r + g^{tr}c_{tkj} + g^{tr}c_{tjk} + 2g_{jk}V^r - 2\delta_j^r V_k) \cdot (c_{irs} + c_{sri} + c_{sir} + 2g_{ir}V_s - 2g_{is}V_r) - \\ &-\frac{1}{2}c_{ij}^r(c_{rks} + c_{skr} + c_{srk} + 2g_{rk}V_s - 2g_{rs}V_k) + \frac{1}{2}C_{ij}^l(\bar{c}_{lks} - \bar{c}_{lsk}), \end{aligned}$$

where $V_k = V^s g_{sk}$, $c_{ijk} = c_{ij}^s g_{sk}$, $\bar{c}_{ijk} = \bar{c}_{ij}^s g_{sk}$.

Theorem 2. *[3, 4] Let (M, g) is a 3-dimensional locally homogeneous (pseudo)Riemannian manifold. Then either (M, g) is locally symmetric (w.r.t. the Levi-Civita connection) or (M, g) is locally isometric to a 3-dimensional Lie group with left-invariant (pseudo)Riemannian metric.*

Theorem 3. *[3, 4] Three dimensional locally symmetric (pseudo)Riemannian manifold is locally isometric to one of the following:*

- *(pseudo)Riemann space form $\mathbb{R}^3$, $\mathbb{S}^3$ or $\mathbb{H}^3$ (with zero, positive or negative sectional curvature, respectively), or*
- *direct product $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$, or*
- *Walker manifold (i.e. manifold with parallel isotropic distribution) with Lorentzian metric g, which amdits a local coordinate system (u_1, u_2, u_3) such that the metric tensor has the form $g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & u_2^2\alpha + u_2\beta(u_3) + \xi(u_3) \end{pmatrix}$, where $\varepsilon = \pm 1$; $\alpha \in \mathbb{R}$; β and ξ are arbitrary smooth functions.*

A classification of three-dimensional locally homogeneous (pseudo)Riemannian manifolds is obtained in the work [5]. Next we will use the numbering from this work. In particular, from this classification follows

Theorem 4. *[5] Let $M = G/H$ is a 3-dimensional locally homogeneous manifold with locally symmetric invariant (pseudo)Riemannian metric. Then there exists a basis $\{e_1, \dots, e_h, u_1, u_2, u_3\}$ in Lie algebra of G, where $\{e_i\}$ and $\{u_i\}$ are bases of $\mathfrak{h}$ and $\mathfrak{m}$ respectively, such that the Lee brackets have the form shown in the following list.*

1. Space forms

3.5.1 ($\mathbb{R}^3$ with Riemannian metric)

$$[e_1,e_2]=e_3,\quad [e_1,e_3]=-e_2,\quad [e_1,u_1]=-u_3,\quad [e_1,u_3]=u_1,$$
$$[e_2,u_1]=-u_2,\quad [e_2,u_2]=u_1,\quad [e_3,u_2]=-u_3,\quad [e_3,u_3]=u_2,$$
$$[e_2,e_3]=e_1.$$

3.4.1 ($\mathbb{R}^3$ with Loretzian metric)

$$[e_1,e_2]=e_2,\quad [e_1,e_3]=-e_3,\quad [e_1,u_1]=u_1,\quad [e_1,u_3]=-u_3,$$
$$[e_2,u_2]=u_1,\quad [e_2,u_3]=u_2,\quad [e_3,u_1]=u_2,\quad [e_3,u_2]=u_3,$$
$$[e_2,e_3]=e_1.$$

3.5.2 ($\mathbb{S}^3$ with Riemannian metric)

$$[e_1,e_2]=e_3,\quad [e_1,e_3]=-e_2,\quad [e_1,u_1]=-u_3,\quad [e_1,u_3]=u_1,$$
$$[e_2,e_3]=e_1,\quad [e_2,u_1]=-u_2,\quad [e_2,u_2]=u_1,\quad [e_3,u_2]=-u_3,$$
$$[e_3,u_3]=u_2,\quad [u_1,u_2]=e_2,\quad [u_1,u_3]=e_1,\quad [u_2,u_3]=e_3.$$

3.4.2 ($\mathbb{S}^3$ with Loretzian metric)

$$[e_1,e_2]=e_2,\quad [e_1,e_3]=-e_3,\quad [e_1,u_1]=u_1,\quad [e_1,u_3]=-u_3,$$
$$[e_2,e_3]=e_1,\quad [e_2,u_2]=u_1,\quad [e_2,u_3]=u_2,\quad [e_3,u_1]=u_2,$$
$$[e_3,u_2]=u_3,\quad [u_1,u_2]=e_2,\quad [u_1,u_3]=-e_1,\quad [u_2,u_3]=-e_3.$$

3.5.3 ($\mathbb{H}^3$ with Riemannian metric)

$$[e_1,e_2]=e_3,\quad [e_1,e_3]=-e_2,\quad [e_1,u_1]=-u_3,\quad [e_1,u_3]=u_1,$$
$$[e_2,e_3]=e_1,\quad [e_2,u_1]=-u_2,\quad [e_2,u_2]=u_1,\quad [e_3,u_2]=-u_3,$$
$$[e_3,u_3]=u_2,\quad [u_1,u_2]=-e_2,\quad [u_1,u_3]=-e_1,\quad [u_2,u_3]=-e_3.$$

3.4.3 ($\mathbb{H}^3$ with Loretzian metric)

$$[e_1,e_2]=e_2,\quad [e_1,e_3]=-e_3,\quad [e_1,u_1]=u_1,\quad [e_1,u_3]=-u_3,$$
$$[e_2,e_3]=e_1,\quad [e_2,u_2]=u_1,\quad [e_2,u_3]=u_2,\quad [e_3,u_1]=u_2,$$
$$[e_3,u_2]=u_3,\quad [u_1,u_2]=-e_2,\quad [u_1,u_3]=e_1,\quad [u_2,u_3]=e_3.$$

2. Direct products

1.3.5 ($\mathbb{S}^2\times\mathbb{R}$)

$$[e_1,u_1]=-u_2,\quad [e_1,u_2]=u_1,\quad [u_1,u_2]=e_1.$$

1.3.6 ($\mathbb{H}^2\times\mathbb{R}$)

$$[e_1,u_1]=-u_2,\quad [e_1,u_2]=u_1,\quad [u_1,u_2]=-e_1.$$

3. Walker manifolds

1.1.1 $[e_1,u_1]=u_1,\quad [e_1,u_2]=-u_2.$

1.1.5 $[e_1,u_1]=u_1,\quad [e_1,u_2]=-u_2,\quad [u_1,u_2]=e_1.$

1.8.1 $[e_1,u_2]=u_1,\quad [e_1,u_3]=u_2.$

1.8.4 $[e_1,u_2]=u_1,\quad [e_1,u_3]=u_2,\quad [u_2,u_3]=e_1.$

1.8.5 $[e_1,u_2]=u_1,\quad [e_1,u_3]=u_2,\quad [u_2,u_3]=-e_1.$

2.21.1 $[e_1,e_2]=e_2,\quad [e_1,u_1]=u_1,\quad [e_1,u_3]=-u_3,$
$[e_2,u_2]=u_1,\quad [e_2,u_3]=u_2.$

Table 1

Invariant metric tensors of three-dimensional locally symmetric (pseudo)Riemannian manifolds

Case	Invariant metric tensor	Restrictions
1.1.1 1.1.5	$\begin{pmatrix} 0 & \alpha_{12} & 0 \\ \alpha_{12} & 0 & 0 \\ 0 & 0 & \alpha_{33} \end{pmatrix}$	$\alpha_{12}\alpha_{33} \neq 0$
1.3.5 1.3.6	$\begin{pmatrix} \alpha_{22} & 0 & 0 \\ 0 & \alpha_{22} & 0 \\ 0 & 0 & \alpha_{33} \end{pmatrix}$	$\alpha_{22}\alpha_{33} \neq 0$
1.8.1 1.8.4 1.8.5	$\begin{pmatrix} 0 & 0 & -\alpha_{22} \\ 0 & \alpha_{22} & 0 \\ -\alpha_{22} & 0 & \alpha_{33} \end{pmatrix}$	$\alpha_{22} \neq 0$
2.21.1 3.4.1 3.4.2 3.4.3	$\begin{pmatrix} 0 & 0 & -\alpha_{22} \\ 0 & \alpha_{22} & 0 \\ -\alpha_{22} & 0 & 0 \end{pmatrix}$	$\alpha_{22} \neq 0$
3.5.1 3.5.2 3.5.3	$\begin{pmatrix} \alpha_{33} & 0 & 0 \\ 0 & \alpha_{33} & 0 \\ 0 & 0 & \alpha_{33} \end{pmatrix}$	$\alpha_{33} \neq 0$

The invariance condition of the metric tensor g has the form:

$$(\psi_i)^t \cdot g + g \cdot \psi_i = 0, \quad i = 1, \dots, h,$$

where ψ_i is a isotropy representation, which is defined via formula $\psi_i(u_j) = [e_i, u_j]$; $(\psi_i)^t$ is a transposed matrix. The form of the invariant metric tensor for each case of the theorem given in the following table.

A classification of three-dimensional Riemannian Lie groups was obtained by J. Milnor in [6]. The classification of three-dimensional Lorentzian Lie groups was obtained in [4, 7, 8].

Theorem 5. *[6] Let G is a 3-dimensional Lie group wuth left-invariant Riemannian metric. Then*

- *if G is a unimodular, then there exists an orthonormal basis $\{e_1, e_2, e_3\}$ in Lie algebra $\mathcal{U}$ of group G such that*

$$[e_1, e_2] = \alpha_3 e_3, \quad [e_1, e_3] = -\alpha_2 e_2, \quad [e_2, e_3] = \alpha_1 e_1.$$

- *if G is a nonunimodular, then there exists an orthonormal basis $\{e_1, e_2, e_3\}$ in Lie algebra $\mathcal{NU}$ of group G such that*

$$[e_1, e_2] = (2 - \alpha_2)e_2 + \alpha_3 e_3, \quad [e_1, e_3] = \alpha_1 e_2 + \alpha_2 e_3,$$

Theorem 6. *[4, 7] Let G is a 3-dimensional Lie group wuth left-invariant Lorentzian metric. Then*
1) if G is a unimodular, then there exists an pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ in Lie algebra of group G such that the metric Lie algebra is contained in the following list

$\mathcal{A}_1$*:* $[e_1, e_2] = \alpha_3 e_3$, $[e_1, e_3] = -\alpha_2 e_2$, $[e_2, e_3] = \alpha_1 e_1$ *with e_1 timelike;*

$\mathcal{A}_2$*:* $[e_1, e_2] = (1 - \alpha_2) e_3 - e_2$, $[e_1, e_3] = e_3 - (1 + \alpha_2) e_2$, $[e_2, e_3] = \alpha_1 e_1$ *with e_3 timelike;*

$\mathcal{A}_3$*:* $[e_1, e_2] = e_1 - \alpha_1 e_3$, $[e_1, e_3] = -\alpha_1 e_2 - e_1$, $[e_2, e_3] = \alpha_1 e_1 + e_2 + e_3$ *with e_3 timelike;*

$\mathcal{A}_4$: $[e_1, e_2] = \alpha_3 e_2$, $[e_1, e_3] = -\alpha_2 e_1 - \alpha_1 e_2$, $[e_2, e_3] = -\alpha_1 e_1 + \alpha_2 e_2$ *with* e_1 *timelike and* $\alpha_2 \neq 0$.

2) if G *is a nonunimodular, then there exists an pseudo-orthonormal basis* $\{e_1, e_2, e_3\}$ *in Lie algebra of group* G *such that the metric Lie algebra is contained in the following list*

$\mathcal{A}$: $[e_1, e_3] = \alpha_1 \sin\alpha_3\, e_1 - \alpha_2 \cos\alpha_3\, e_2$, $[e_2, e_3] = \alpha_1 \cos\alpha_3\, e_1 + \alpha_2 \sin\alpha_3\, e_2$ *with* e_3 *timelike and* $\sin\alpha_3 \neq 0$, $\alpha_1 + \alpha_2 \neq 0$, $\alpha_1 \geqslant 0$, $\alpha_2 \geqslant 0$;

$\mathcal{B}$: $[e_1, e_3] = \alpha_3 e_1 - \alpha_4 e_2$, $[e_2, e_3] = \alpha_1 e_1 + \alpha_2 e_2$ *with nonzero* $\langle e_2, e_2\rangle = -\langle e_1, e_3\rangle = 1$ *and* $\alpha_2 \neq \alpha_3$;

$\mathcal{C}_1$: $[e_1, e_3] = \alpha_3 e_1 + \alpha_1 e_2$, $[e_2, e_3] = \alpha_1 e_1 + \alpha_2 e_2$ *with* e_2 *timelike and* $\alpha_2 \neq \alpha_3$;

$\mathcal{C}_2$: $[e_1, e_3] = \alpha_2 e_1 - \alpha_3 e_2$, $[e_2, e_3] = \alpha_1 e_1 + \alpha_2 e_2$ *with* e_2 *timelike and* $\alpha_2 \neq 0$, $\alpha_1 + \alpha_3 \neq 0$.

2. Results

In these notations, the following holds.

Theorem 7. *Let* $(G/H, g)$ *be a 3-dimensional locally symmetric (pseudo)Riemannian manifold with metric connection with invariant vectorial torsion. If the curvature tensor is zero, then*

- *in cases 1.1.1, 1.8.1, 2.21.1, 3.4.1, 3.5.1 vector* V *is trivial;*
- *in case 1.1.5 vector* V *has coordinates* $\left(0, 0, \pm\frac{1}{\sqrt{-\alpha_{12}\alpha_{33}}}\right)$;
- *in case 1.3.5 vector* V *has coordinates* $\left(0, 0, \pm\frac{1}{\sqrt{-\alpha_{22}\alpha_{33}}}\right)$ *and the metric tensor must be Lorentzian;*
- *in case 1.3.6 vector* V *has coordinates* $\left(0, 0, \pm\frac{1}{\sqrt{\alpha_{22}\alpha_{33}}}\right)$ *and the metric tensor must be Riemannian;*
- *in case 1.8.4 vector* V *has coordinates* $\left(\pm\frac{1}{\alpha_{22}}, 0, 0\right)$.

In cases 1.8.5, 3.4.2, 3.4.3, 3.5.2, 3.5.3 curvature tensor cannot be zero.

Theorem 8. *Let* (G, g) *be a 3-dimensional metric Lie group with metric connection with invariant vectorial torsion. If the curvature tensor is zero, then*
in case $\mathcal{U}$: $V = 0$ *and*

- *or* $\alpha_1 = 0$, $\alpha_2 = \alpha_3$;
- *or* $\alpha_2 = 0$, $\alpha_1 = \alpha_3$;
- *or* $\alpha_3 = 0$, $\alpha_1 = \alpha_2$;

in case $\mathcal{NU}$: $\alpha_2 = 1 \pm \sqrt{1 - \alpha_1^2}$, $\alpha_1 = \alpha_3$ *and*

- *or* $V = (-2, 0, 0)$;
- *or* $V = \left(0, -\frac{\left(1\pm\sqrt{1-\alpha_1^2}\right)V^3}{\alpha_1}, V^3\right)$, $V^3 = \sqrt{\pm 2\sqrt{1-\alpha_1^2} - 2}$

in case $\mathcal{A}_1$: $V = 0$ *and*

- *or* $\alpha_1 = 0$, $\alpha_2 = \alpha_3$;
- *or* $\alpha_2 = 0$, $\alpha_1 = -\alpha_3$;
- *or* $\alpha_3 = 0$, $\alpha_1 = -\alpha_2$;

in case $\mathcal{A}_2$: $\alpha_1 = 0$, $\alpha_2 = 0$ *and* $V = 0$;
in case $\mathcal{A}_3$: $\alpha_1 = 0$ *and*

- *or* $V = (0, 1, 1)$;

- *or* $V = (0, -2, -2)$*;*

in case $\mathcal{A}_4$ *curvature tensor cannot be zero; in case* $\mathcal{A}$*:*

- *or* $\alpha_1 = 0$*,* $\alpha_3 = \pm\pi/2$*,* $V = (\pm\alpha_2, 0, 0)$*;*
- *or* $\alpha_1 = 0$*,* $\alpha_3 = \pm\pi/2$*,* $V = (0, 0, \mp\alpha_2)$*;*
- *or* $\alpha_2 = 0$*,* $\alpha_3 = \pm\pi/2$*,* $V = (0, \pm\alpha_1, 0)$*;*
- *or* $\alpha_2 = 0$*,* $\alpha_3 = \pm\pi/2$*,* $V = (0, 0, \mp\alpha_1)$*;*
- *or* $\alpha_1 = \alpha_2$*,* $V = (0, 0, -\alpha_1 \sin(\alpha_3))$*;*

in case $\mathcal{B}$*:* $\alpha_4 = 0$ *and*

- *or* $V = (\alpha_2 - \alpha_3, 0, 0)$*;*
- *or* $V = (-\alpha_2, 0, 0)$*;*

in case $\mathcal{C}_1$*:* $\alpha_1 = 0$ *and*

- *or* $\alpha_2 = 0$*,* $V = (0, 0, \alpha_3)$*;*
- *or* $\alpha_3 = 0$*,* $V = (0, 0, \alpha_2)$*;*
- *or* $\alpha_2 = 0$*,* $V = (0, \pm\alpha_3, 0)$*;*

in case $\mathcal{C}_2$ *curvature tensor cannot be zero.*

Remark 1. *Thus, in dimension 3, all metric connections with vector torsion, for which locally homogeneous (pseudo) Riemannian manifolds are flat, are defined.*

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